Moments conservation in adaptive Vlasov solver

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An adaptive semi-Lagrangian method for the Vlasov equation based on a multiresolution analysis with moments conservation

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Abstract. We previously developed an adaptive semi-Lagrangian solver using a multiresolution analysis based on interpolets which are a kind of interpolating wavelets introduced by Deslauriers and Dubuc. This paper introduces a new multiresolution approximation for this solver which allows to conserve moments up to any order by using the lifting method introduced by Sweldens.
Keywords. Vlasov; phase-space grid; adaptive; multiresolution; plasma physics; beam physics.

The model we consider throughout this paper is the nonrelativistic Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0, \tag{1}$$

where the self electric field $\mathbf{E}$ is computed from Poisson's equations. The magnetic field is external and considered to be known.

The numerical resolution of the Vlasov equation is usually performed by particle methods (PIC) which is known to developp an inherent numerical noise in some cases. To remedy to this problem an obtain a more accurate description of the distribution function, methods discretizing the Vlasov equation on a mesh of phase space have been proposed [2, 5, 6]. In order to avoid the high numerical cost of such methods using a uniform and fixed mesh, we develop adaptive methods.

In the present work, the adaptive method is overlayed to a classical semi-Lagrangian method based on the conservation of the distribution function along particle trajectories. Furthermore, we have chosen to introduce a phase-space mesh which can be refined or derefined adaptively in time. For this purpose, we use a technique based on multiresolution analysis which is in the same spirit as the methods developed in particular by S. Bertoluzza [1], A. Cohen *et al.* [3] and M. Griebel and F. Koster [4]. We represent the distribution function on a wavelet basis at different scales. We can then compress it by eliminating coefficients which are small and accordingly remove the associated mesh points (section 1).

Here, we particularly want that the distribution fonction satisfies physical properties as mass conservation or kinetic energy conservation. To this purpose, we explain (section 2) how to use the lifting procedure introduced by W. Sweldens [7] in order to ensure the conservation of moments of (1) .

Another specific feature of our method is that we use an advection in physical and velocity space forward in time to predict the useful grid points for the next time step, rather than restrict ourselves to the neighboring points. This enables us to use a much larger time step, as in the semi-Lagrangian method the time step is not limited by a Courant condition. Once the new mesh is predicted, the semi-Lagrangian methodology is used to compute the new values of the distribution function at the predicted mesh points, using an interpolation based on the wavelet decomposition of the old distribution function. The mesh is then refined again by performing a wavelet transform, and eliminating the points associated to small coefficients. The numerical algorithm as well as relevant numerical results are presented in the two last sections.

1 Multiresolution analysis

The semi-Lagrangian method consists in computing point values of the distribution function on a grid of phase-space. It consist of two steps, an advection step need to determine the origin of the characteristics ending at the grid points and an interpolation step which is used to compute the value of the distribution function at these points.

The multiresolution approximation will be used to minimize the number of interpolation points for a given approximation error.

First, we introduce the change of scale using a Lagrange interpolating polynomial of any odd degree. For any value of $j \in \mathbb{Z}$, we consider a uniform grid G^j of step 2^{-j}. The grid points are located at $x_k^j = k2^{-j}$. This defines an infinite sequence of grids that we denote by $(G_j)_{j\in\mathbb{Z}}$, and j will be called the level of the grid.

In order to map a function of values $(c_k^j)_{k\in\mathbb{Z}}$ defined on the grid, from one level j to the next finer $j+1$ and reciprocally, we define respectively a projection operator (merely a restriction operator) and a prediction operator using the terminology in [3]

$$\begin{aligned} P_{j+1}^j : G_{j+1} \to G_j, \qquad & P_j^{j+1} : G_j \to G_{j+1}, \\ c_{2k}^{j+1} \mapsto c_k^j, \qquad & \text{such that } c_{2k}^{j+1} = c_k^j, \\ & c_{2k+1}^{j+1} = P_{2N-1}(x_{2k+1}^{j+1}) = \sum_{n=1-N}^{N} a_n c_{k+n}^j, \end{aligned}$$

where P_{2N-1} stands for the Lagrange interpolation polynomial of odd degree $2N-1$ centered at the point (x_{2k+1}^{j+1}). Hence, we can define a grid function f_{j+1} defined on G_{j+1} by the sequence $(c_k^{j+1})_{k\in\mathbb{Z}}$ or equivalently by the sequences $(c_k^j)_{k\in\mathbb{Z}}$ and $(d_k^j)_{k\in\mathbb{Z}}$ where

$$c_k^j = c_{2k}^{j+1} \quad \text{and} \quad d_k^j = c_{2k+1}^{j+1} - P_{2N-1}(x_{2k+1}^{j+1}) = c_{2k+1}^{j+1} - \sum_{n=1-N}^{N} a_n c_{k+n}^j = c_{2k+1}^{j+1} - \sum_{n=1-N}^{N} a_n c_{2k+2n}^{j+1}. \quad (2)$$

For a better theoretical understanding, we shall cast these change of scale relations into the framework of biorthogonal wavelets introduces by Cohen, Daubechies and Feauveau [**?**]. In this framework, the two dual scaling functions φ and $\tilde{\varphi}$ are uniquely defined by the sequences $(h_n)_{n\in\mathbb{Z}}$ and $(\tilde{h}_n)_{n\in\mathbb{Z}}$ through the following scaling relations

$$\varphi(x) = \sum_{n\in\mathbb{Z}} h_n \varphi(2x-n), \qquad \tilde{\varphi}(x) = \sum_{n\in\mathbb{Z}} \tilde{h}_n \tilde{\varphi}(2x-n), \quad (3)$$

and are related through the biorthogonality relation

$$\int \varphi(x) \cdot \tilde{\varphi}(x-k)\, dx = \delta_{0,k}. \quad (4)$$

To the couple of scaling functions $(\varphi, \tilde{\varphi})$, we can associate a couple of wavelets $\psi, \tilde{\psi}$ defined by

$$\psi(x) = \sum_{n\in\mathbb{Z}} g_n \varphi(2x-n) \text{ with } g_n = (-1)^{n+1}\tilde{h}_{1-n}, \qquad \tilde{\psi}(x) = \sum_{n\in\mathbb{Z}} \tilde{g}_n \tilde{\varphi}(2x-n) \text{ with } \tilde{g}_n = (-1)^{n+1} h_{1-n}. \quad (5)$$

We then denote by $\varphi_k^j = \varphi(2\cdot - k)$ and the same for $\tilde{\varphi}_k^j$, ψ_k^j and $\tilde{\psi}_k^j$. Sequences for $j \in \mathbb{Z}$ of space V_j and $\tilde{V}_j$ repectively spanned by $(\varphi_k^j)_{k\in\mathbb{Z}}$ and $(\tilde{\varphi}_k^j)_{k\in\mathbb{Z}}$ define a multiresolution approximation. Morever, the following decompositions hold (see [**?**] for a proof)

$$V_{j+1} = V_j \oplus W_j, \qquad \tilde{V}_{j+1} = \tilde{V}_j \oplus \tilde{W}_j.$$

where spaces of details W^j and $\tilde{W}_k^j$ are respectively spanned by $(\psi_k^j)_{k\in\mathbb{Z}}$ and $(\tilde{\psi}_k^j)_{k\in\mathbb{Z}}$.

Based on this decomposition, a function $f_{j+1} \in V_{j+1}$ can be expressed on either basis

$$f_{j+1} \;=\; \sum_k c_k^{j+1}\varphi_k^{j+1} \;=\; \sum_k c_k^j \varphi_k^j + \sum_k d_k^j \psi_k^j. \quad (6)$$

Using the biorthogonality relation (4) and the second scaling relation in (3), we can express for a general couple of biorthogonal wavelets

$$c_k^j = \sum_n \tilde{h}_{n-2k} c_n^{j+1}, \qquad d_k^j = \sum_n \tilde{g}_{n-2k} c_n^{j+1}. \tag{7}$$

Comparing the above relations with the decomposition (2) we get for our multiresolution analysis based on Lagrange interpolating polynomials, we find the filter sequences defining our scaling functions and wavelets:

$$\tilde{h}_n = \delta_{0,n} \quad \text{and} \quad \tilde{g}_{2n+1} = \delta_{0,n},\ \tilde{g}_{2n} = -a_n. \tag{8}$$

In table 1 below, we give the values of a_n for interpolating polynomials P_{2N-1} of degree 1 or 3 that is $N = 1$ or 2.

$N = 1:\quad a_0 = a_1 = \frac{1}{2},\ a_n = 0 \quad else; \qquad N = 2:\quad a_{-1} = a_2 = -\frac{1}{16},\ a_0 = a_1 = \frac{9}{16},\ a_n = 0 \quad else.$

Table 1: Coefficients of interpolating polynomial for different values of N

Note that the dual wavelet is independent of the interpolating polynomial and we easily deduce from the values of $(\tilde{h}_n)_n$ that $\tilde{\varphi}(0) = 1$ and $\tilde{\varphi}(x) = 0$ for $x \neq 0$. This function is obviously not in $L^2(\mathbb{R})$. However the biorthogonal wavelets can still be defined in the dual setting $C^0(\mathbb{R})$ and Radon measures. As $g_n = (-1)^{n+1}\tilde{h}_{1-n}$, we have $g_n = \delta_{1,n}$ and thus using (5)

$$\psi(x) = \phi(2x - 1).$$

The scaling function φ and wavelet ψ for these values of N are displayed in Figures 1 and 2.

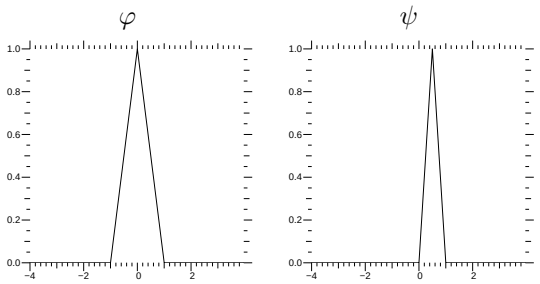

Figure 1: Scaling functions and wavelets for linear interpolation polynomials ($N = 1$.)

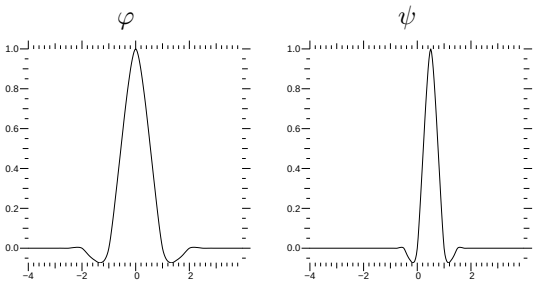

Figure 2: Scaling functions and wavelets for cubic interpolation polynomials ($N = 2$.)

We now consider the thresholding. The decomposition of a function $f_{j+1} \in V_{j+1}$ on the coarser grid V_j and its supplementary W_j in V_{j+1} allows us to isolate in W_j the improvement of approximation in V_{j+1} with respect to V_j. Hence large details d_k^j mean that the improvement is localy important around grid point $2^{-j}(2k+1)$ and conversely small details mean that the local improvement is small. Hence the representation of a function of V_{j+1} can be compressed with a controled approximation loss by setting to 0 the details with an absolute value less than some given threshold ϵ_j depending on the level j.

In decomposition formula 6, we consider terms $d_k^j \psi_k^j$. Let $\epsilon_j > 0$ be a given threshold, then we have

$$\|d_k^j \psi_k^j\|_{L^p} = |d_k^j| 2^{-\frac{j}{p}} \|\psi\|_{L^p}, \quad \text{for } p \in \mathbb{N}^* \quad \text{and} \quad \|d_k^j \psi_k^j\|_{L^\infty} = |d_k^j| \|\psi\|_{L^\infty}.$$

Assuming that all details smaller than the threshold are eliminated from the expression of f_{j+1}, we get a new approximation

$$f_{j+1}^s = \sum_k c_k^j \varphi_k^j + \sum_{k | d_k^j \geq \epsilon_j} d_k^j \psi_k^j,$$

and the error commited because of this thresholding process is

$$e_{j+1}^s = \sum_{k,\ d_k^j < \epsilon_j} d_k^j \psi_k^j.$$

Assuming that f_{j+1} has compact support, the number of removed terms is finite and hence we can compute the following bound for the error:

$$e^s_{j+1} \quad \leq \quad \epsilon_j 2^{-\frac{j}{p}} \|\psi\|_{L^p} \, Card(\{k| \ |d^j_k| < \epsilon_j\}). \tag{9}$$

2 Conservation of moments

The thresholding procedure consists in removing terms from W_j, i.e. linear combinations of the ψ^j_k. Hence this procedures conserves moments if the corresponding moments of ψ, $\int x^p \psi(x)\, dx$ vanish. When numerically solving the Vlasov equation, it is often very important to conserve the number of particles and the kinetic energy.

We saw that the wavelet associated to the interpolating scaling function is $\psi(x) = \varphi(2x-1)$. Hence, as the moments of φ do not vanish, the moments of ψ do not vanish. However, the lifting procedure introduced by Sweldens [7] can be used to define a new set of biorthogonal scaling functions and wavelets from a given one so that some desired properties are satisfied.

Consider the decomposition $V_{j+1} = V_j + W_j$. Following Sweldens' idea, we can modify W_j without modifying the multiresolution approximation $(V_j)_j$ by taking a new wavelet $\bar{\psi}$ of the following form

$$\bar{\psi} = \psi - \sum_k s_k \, \varphi(x-k)$$

where the coefficients $(s_k)_k$ define the new wavelet.

To conserve the mass, one needs that $\int \bar{\psi}\, dx = 0$ which yields

$$0 \quad = \quad \int \varphi(2x-1)\, dx - \sum_k s_k \int \varphi(x-k)\, dx \quad = \quad \left(\frac{1}{2} - \sum_k s_k\right) \int \varphi(x) dx$$

Then we need to ensure that $\sum_k s_k = 1/2$. To keep the symmetry of $\bar{\psi}$, we also impose that $s_k = s_{1-k}$. Then the simplest choice for $(s_k)_k$ is

$$s_0 = s_1 = 1/4 \quad \text{and} \quad s_k = 0 \quad \text{else.}$$

With this choice of s_0 and s_1, $\bar{\psi}$ is an even function so that the first order moment is also conserved.

More generally, Using the scaling relation 3 which allows to compute a wavelet at any level from the wavelets at the next finer level, we can compute the s_k in order to conserve the moment of any even order, always keeping the symmetry ensuring $s_k = s_{1-k}$. Then if all the moments of even order are conserved up to some even n, all the moments of any odd order are also conserved up to $n+1$.

In figure 3 and table 2, we respectively construct the corrected wavelet in order to ensure mass conservation and give the simplest choice of s_k in order To ensure the conservation of the second order moment.

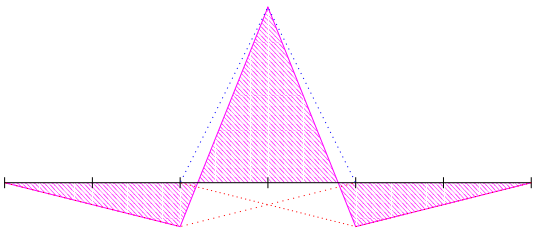

Figure 3: How to correct linear wavelet in order to conserve zero order moment.

linear wavelets:	$s_1 = \frac{19}{64}$	and	$s_2 = \frac{-3}{64}$
cubic wavelets:	$s_1 = \frac{149}{512}$	and	$s_2 = \frac{-21}{512}$

Table 2: Simplest choice of s_k for second order moment conservation.

3 The adaptive algorithm

In the initialization phase, we first compute the wavelet decomposition of the initial condition f_0 which is known analytically, and then proceed by compressing it, i.e. eliminating the details which are smaller

than a threshold that we set. We then construct an adaptive mesh which, from all the possible points at all the levels between our coarsest and finest, contains only those of the coarsest and those corresponding to details which are above the threshold. We denote by $\tilde{G}$ this mesh.

- **Prediction in x:** We predict the positions of points where the details should be important at the next time split step by advancing in x the characteristics originating from the points of the mesh $\tilde{G}$. Then we retain the grid points, at one level finer as the starting point, the support of which contains the end point the characteristic.
- **Construction of mesh $\hat{G}$:** From the predicted mesh $\tilde{G}$, we construct the mesh $\hat{G}$ where the values of the distribution at the next time step shall be computed. This mesh $\hat{G}$ contains exactly the points necessary for computing the wavelet transform of f^* at the points of $\tilde{G}$.
- **Advection in x:** We compute the origin of the characteristics for each point of $\hat{G}$ and interpolate the value of f^* at this point at the level where it arrives.
- **Wavelet transform of f^*:** We compute the c_k and d_k coefficients at the points of $\tilde{G}$ from the values of f^* at the points of $\hat{G}$.
- **Compression:** We eliminate the points of $\tilde{G}$ where the details d_k are lower than the fixed threshold.
- **Computation of the electric field:** We compute ρ on the finest grid and solve Poisson.
- Same steps for the v part of the splitting method.

4 Numerical results

4.1 Semi-Gaussian beam in periodic focusing channel

In order to assess the benefits of the adaptive solver we computed the transverse evolution of a semi-Gaussian beam in and periodic focusing channel. For such a beam the initial distribution function reads

$$f(r,v) = \frac{I}{\pi a^2\sqrt{2\pi}b} e^{-\frac{1}{2}(v^2/b^2)} \text{ if } r < a, \quad \text{ and } f(r,v) = 0 \text{ else.}$$

The periodic focusing field is of the form $\alpha(1+\cos 2\pi z/S)$ for a tune depression σ/σ_0 of 0.17.

Figure 4 represents snapshots of the evolution along, with the grid points kept by the adaptive algorithm for the computation. We notice that the adaptive grid follows very well the evolution of the fine structures.

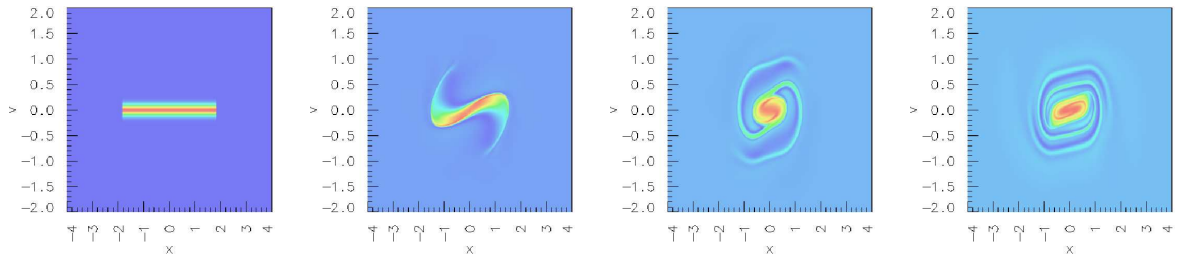

Figure 4: Semi-Gaussian beam and the associated adaptive grid in a periodic focusing channel.

4.2 The two stream instability

The initial function is given by:

$$f_0 = \frac{1}{\sqrt{2\pi}} v^2 \exp(-v^2/2)(1+\alpha cos(kx))$$

with $\alpha = 0.05$ and $k = 0.5$. The physical Mesh is $x-$periodic : $[0, 4\pi[\times[-6, 6[$.

In Figure 5, we compare computations of the distribution function with and without ensuring conservation of mass. We notice that in the case with mass conservation, unstability problems which occur in the other case are avoided. We also notice that the numerical diffusion is more important when we use the lifting procedure.

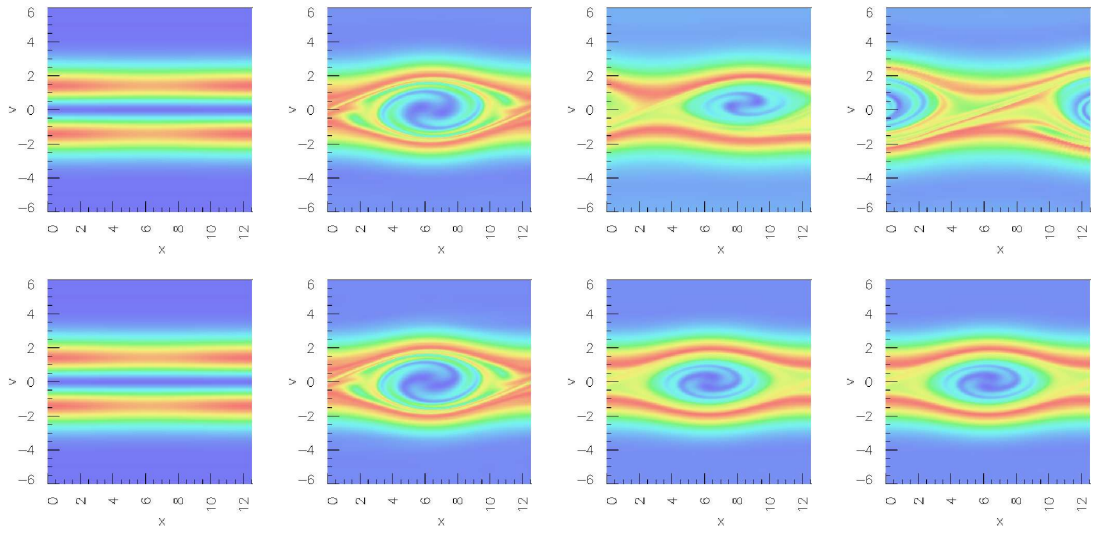

Figure 5: Evolution of a two stream instability without (top) and with (bottom) mass conservation.

5 Conclusion

We have introduced in this paper the concept of an adaptive semi-Lagrangian Vlasov solver and proved the feasability of the method for 1D model problems. The method we proposed can be generalized to higher dimensions. Adaptivity is defined through a multiresolution analysis based on interpolating wavelets which is coupled very naturally to the semi-Lagrangian method. Grid points are added on a finer grid only where interpolation on the coarser grid does not do a good enough job.

Adaptive methods enable semi-Lagrangian methods not to waste computing time in regions where nothing is happening and thus make them a lot more efficient. On the other hand, an important overhead is introduced, and programming such a method in an efficient way is a lot more challenging. We are still working on it.

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